Chapter 8
Survey Adjustments for Conventional Surveys

8-1. General

The adjustment of survey and photogrammetric data is a critical component in the determination of reliable coordinates, directions, distances, and elevation data. The adjustment technique and the analysis of the estimated parameters should be given the same priority as the data collection and recording procedures. An adjustment is a method of dealing with redundant data. Redundancy can be considered excess information. If the redundancy of a system is zero, the system of observations would not warrant an adjustment. For example, if a surveyor measured the distance from two known survey points to an unknown location (range - range intersection), the two distances from the known points would uniquely determine the two-dimensional location of the unknown point. In this example there is no extra or redundant data, therefore, an adjustment would not be warranted.

a. Survey computations, whether made on a local system or a standardized accepted system (e.g., SPCS), are for all intents and purposes identical. The adjustment of raw survey data is treated as independent observations and adjusted as part of a total network. A variety of methods may be used to adjust the survey data, including compass (or Bowditch) rule, transit rule, the Crandall Method, and the method of least squares. This chapter describes some of the methods used to perform horizontal and vertical adjustments and provides guidance in evaluating the adequacy and accuracy of the adjustment results.

b. Differential carrier phase GPS survey observations are adjusted no differently than conventional surveys. Each three-dimensional GPS baseline vector is treated as a separate distance observation and adjusted as part of a trilateration network. A variety of the techniques developed in this chapter can be used to adjust observed GPS baselines to fit existing control. However, they are usually adjusted by least squares. Refer to EM 1110-1-1003 for further guidance on GPS baseline adjustment.

8-2. Adjustment Methods

An adjustment may involve a mean of observations, balancing of a traverse, and an adjustment by least squares. This chapter will briefly address traverse balancing and the concept of a weighted mean. The main scope of the chapter will deal with the adjustment of data using the method of least squares. The method of least squares is the most prevalent adjustment technique utilized by commercial software packages.

8-3. Traverse Adjustment (Balancing)

The method of traverse adjustment is widely used by the land surveying and engineering communities. This method of adjustment is easy to perform and provides adequate results for many survey applications. The method of traverse adjustment depends on the precision of the directions as compared to the precision of the distances. The equations necessary for each adjustment method are available in any elementary survey text.

a. Crandall rule. The Crandall rule is used when the angular measurements (directions) have greater precision than the linear measurements (distances). This method allows for the weighting of measurements and has properties similar to the method of least squares adjustment. Although the technique provides adequate results, it is seldom utilized because of its complexity. Also, with the advent of the personal computer, a traditional least squares adjustment can be performed with little effort.

b. Transit rule. The transit rule is utilized when the angular measurements are of greater precision than the linear measurements. For example, if a surveyor was using a transit or theodolite for angular measurements and stadia for linear measurements, the transit rule adjustment would be applicable. This method is rarely used because modern distance measuring equipment (DME) and electronic theodolites provide distance and angular measurements with equal precision.

c. Compass rule. The compass rule adjustment is used when the angular and linear measurements are of equal precision. This is the most widely used traverse adjustment method. Since the angular and linear precision are considered equivalent, the angular error is distributed equally throughout the traverse. For example, the sum of the interior angles of a five-sided traverse should equal 540° 00’ 00".0, but if the sum of the measured angles equals 540° 01’ 00".0, a value of 12".0 must be subtracted from each observed angle to balance the angles within traverse. After balancing the angular error, the linear error is computed by determining the sums of the north-south latitudes and east-west departures. The misclosure in latitude and departure is applied proportional to the distance of each line in the traverse.

d. Example compass rule adjustment. A four-sided closed traverse was performed (Figure 8-1) using a twenty-second (20") theodolite and a one-hundred-foot (100') steel tape. The linear and angular measurements were considered of equal precision, therefore, the compass rule was utilized to adjust the traverse. The observed and adjusted angles, azimuths, latitudes, and departures are listed in Table 8-1. The following four steps demonstrate how a compass rule adjustment is performed for a loop traverse.

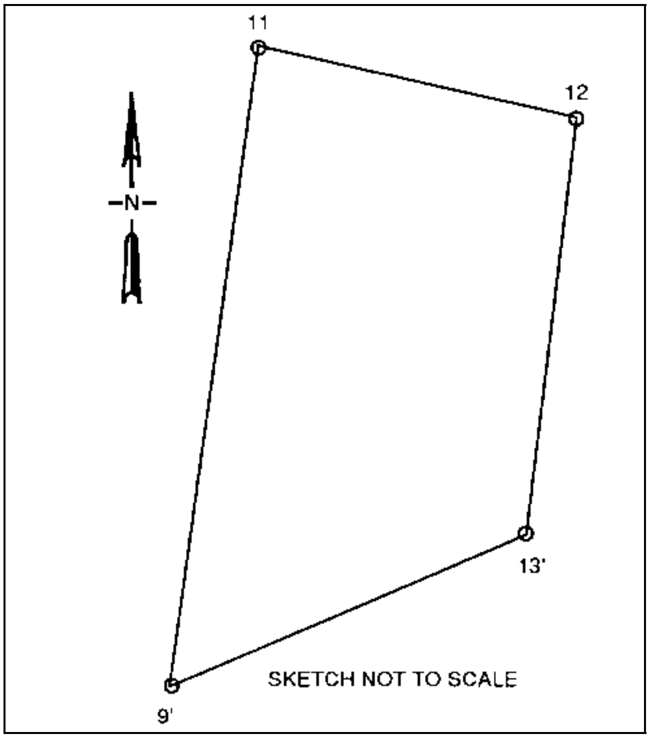

Figure 8-1. Loop traverse

Step 1. The angular error (α_e) of the polygon is computed by differencing the measured (α_m) and true (α_t) angular closure. The measured angular closure is the summation of the interior or exterior horizontal angles in the traverse. If interior angles were measured when performing a loop traverse, the true angular closure equals:

$$\alpha_t = (n-2) * 180^o$$

n = number of sides in the traverse

α_t = true angular closure

If exterior angles were measured when performing a loop traverse, the true (α_t) angular closure would equal:

$$\alpha_t = (n+2) * 180^o$$

n = number of sides in the traverse

α_t = true angular closure

Angular Error

$$\alpha_e = (\alpha_m - \alpha_t)$$

α_e = angular error

The angular error (α_e) is divided by the number of sides (n) in the traverse and is distributed equally to all of the measured angles. If the angular error is negative, the error is added to all the angles in the traverse. If the angular error is positive, the error is subtracted from all of the angles in the traverse. After the angular misclosure has been distributed throughout the traverse, the summation of the interior or exterior angles should equal the true angular closure (α_t). The four-sided closed traverse in Figure 8-1 and Table 8-1 has an angular error (α_e) of four arc seconds (4"). In this example, the angular error is negative; therefore, to balance the angles within the traverse, one arc second (1") must be added to each angle in the traverse.

Step 2. The horizontal angles are converted to bearings or azimuths. To compute the "true" bearing or azimuth of each line in the traverse, one known bearing or azimuth must be available prior to the adjustment process. If a known bearing or azimuth is not available, a "false" bearing can be used to perform the adjustment process. If a "false" bearing is utilized, the traverse will be oriented relative to this bearing. The example in Figure 8-1 and Table 8-1 has a known azimuth between stations 11 and 12. The azimuth for each line in the traverse was computed by adding the back azimuth between stations and the angle to the right. The back azimuth is computed by adding or subtracting one-hundred and eighty degrees (180°) from the forward azimuth. The angle right equals the measured interior angle.

Back Azimuth

If the forward azimuth $\alpha_{ab} < 180^o$

The backward azimuth $\alpha_{ba} = \alpha_{ab} + 180^o$

If the forward azimuth $\alpha_{ab} > 180^o$

The backward azimuth $\alpha_{ba} = \alpha_{ab} - 180^o$

Table 8-1
Compass Rule Adjustment

Station	Measured Angle	Balanced Angle	Azimuth	Horiz. Distance	Unadjusted Latitude	Unadjusted Depart	Adjusted Latitude	Adjusted Depart	Coordinates	Adjusted Length	Adjusted Direction
12			103°-03'-14"	110.84'							
11	85°-05'-33"	85°-05'-34"	188°-03'-48"	219.51'	-217.29'	-31.11'	-217.30'	-31.12'			
9'	58°-48'-39"	58°-48'-40"	66°-57'-28"	130.05'	50.90'	119.67'	50.90'	119.66'			
13'	120°-52'-29"	120°-52'-30"	7°-49'-58"	142.70'	141.37'	19.45'	141.36'	19.44'			
12	95°-13'-15"	95°-13'-16"	288°-03'-14"	110.84'	25.04'	-107.98'	25.04'	-107.98			
		Σ = 360°									
$\Sigma\alpha_m$ =	359°-59'-56"										
α_t =	360°										
α_e =	-4"										
			D =	Σl_{ij}	$\Sigma\Delta N_m$	$\Sigma\Delta E_m$	$\Sigma\Delta N$	$\Sigma\Delta e$			
			SUM	603.10'	0.02'	0.03'	0	0			

(ρ) Line of closure = 0.036' Area = 20,081 Square Feet

(P) Precision = 1/16,726 Area = 0.46 Acres

Adjustment by BAF Rule COMPASS RULE

(ρ) Line of closure $\left(\rho = \sqrt{\sum N_m^2 + \sum E_m^2}\right)$

(P) Traverse precision $\left(\frac{P}{D}\right)$

Step 3. The latitude and departure for each course in the traverse are computed using the bearing or azimuth and distance of the line. If bearings are utilized, the latitude and departure must be identified as negative or positive. The latitude of a line increases south to north. The departure of a course increases from west to east. In the example from Figure 8-1, the line from station 13' to 12 would have a positive latitude and departure. The line from station 12 to 11 would have a positive latitude and negative departure. When azimuths are utilized, the algebraic sign (+/-) of the latitude and departure is accounted for in the trigonometric functions.

Latitude and Departure

$$\Delta N_m = \cos(BRG) * l_{ij}$$

$$\Delta E_m = \sin(BRG) * l_{ij}$$

or

$$\Delta N_m = \cos(\alpha) * l_{ij}$$

$$\Delta E_m = \sin(\alpha) * l_{ij}$$

where

ΔN_m = measured latitude

ΔE_m = measured departure

BRG = bearing

α = azimuth

l_{ij} = distance between stations i and j

Step 4. The summation of the measured latitudes ($\Sigma\Delta N_m$) and departures ($\Sigma\Delta E_m$) represents the error in northing and easting of the traverse. The northing and easting error is distributed throughout the traverse proportional to the distance of each line (l_{ij}) and the perimeter distance of the traverse (D). The precision of the traverse (P) is equal to the line of closure (ρ) divided by the perimeter distance of the traverse.

Latitude and Departure Error

$$\delta N_e = (l_{ij}/D) * \Sigma\Delta N_m$$

$$\delta E_e = (l_{ij}/D) * \Sigma\Delta E_m$$

Traverse Precision

$$P = \rho/D$$

where

P = traverse precision

$\rho = ((\Sigma\Delta N)^2 + (\Sigma\Delta E)^2)^{0.5}$ (line of closure)

$D = \Sigma\, l_i$

$\Sigma\Delta N_m$ = summation of the measured northings within the traverse

$\Sigma\Delta E_m$ = summation of the measured eastings within the traverse

$\Sigma\, l_i$ = summation of the distances within the traverse

l_{ij} = Distance from station i to station j.

The latitude (δN_e) and departure (δE_e) errors are added or subtracted from the measured latitudes (ΔN_m) and departures (ΔE_m). To obtain the adjusted latitudes (ΔN) and departures (ΔE), the algebraic signs (+/-) of the latitude (δN_e) and departure (δE_e) errors are reversed and the errors are added to the measured latitudes (ΔN_m) and departures (ΔE_m) (Table 8-1).

Adjusted Latitudes and Departures

$$\Delta N = \delta N_m \;+/-\; \delta N_e$$

$$\Delta E = \delta E_m \;+/-\; \delta E_e$$

e. Error detection. Errors in a balanced traverse are difficult to locate. Determining angular and linear errors in a balanced traverse requires inspection of the traverse plot and the line of closure (ρ). The perpendicular bisector of the line of closure (ρ) is utilized to find the angular error. If an angular error exists within the traverse, the perpendicular bisector of the line of closure will "point" (Figure 8-2) to the possible station that contains the error. The perpendicular bisector of the line of closure in Figure 8-2 represents a possible angular error in station A. The line of closure (ρ) may parallel a line within the traverse that may have a linear (distance) error (Figure 8-2). Figure 8-2 represents a possible distance error between stations A and B. These techniques in identifying angular and linear errors are effective when only one error exists within the traverse.

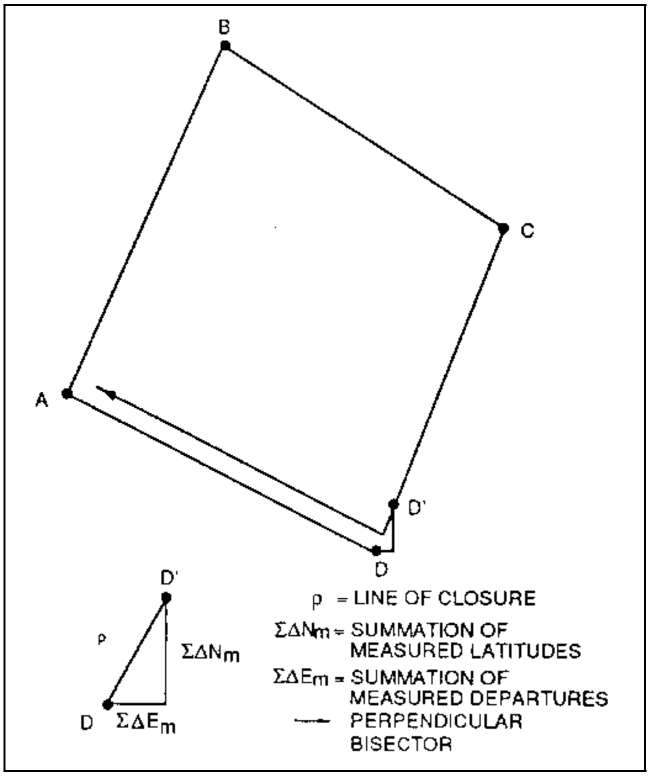

Figure 8-2. Error detection

8-4. Weighted Mean

The weighted mean (X_w) of a set of observations allows the surveyor to estimate the mean and variance of a parameter from a set of independent observations. The weight (w) of an observation is proportional to the inverse of the variance (σ^2) of the observation.

Observational Weight

$$w \approx 1/\sigma^2$$

Weighted Mean

$$X_w = (X_1w_1+X_2w_2+X_3w_3+X_nw_n)/(w_1+w_2+w_3+w_n)$$

X_w = weighted mean

X_1, X_2, X_n = independent observations

w_1, w_2, w_n = observational weights

If the position of a hydrographic vessel was determined by the two techniques Differential Global Positioning System (DGPS) and range azimuth techniques, the concept of the weighted mean can be utilized to determine the most probable position (Table 8-2). The precision of the weighted mean equals the inverse of the sum of the observational weights.

$$\sigma^2 = 1/(w_1 + w_2 + ... + w_n)$$

8-5. Least Squares Adjustment

The method of least squares is the procedure of adjusting a set of observations that constitute an over-determined model (redundancy $\succ$ 0). A least squares adjustment relates the mathematical (functional model) and stochastic (stochastic model) processes that influence or affect the observations. Stochastic refers to the statistical nature of observations or measurements. The least squares principle relies on the condition that the sum of the squares of the residuals approaches a minimum.

$$v^twv = \phi$$

v = observation residual

w = weight of observation

ϕ = minimized criteria

The residuals (v) are the corrections to the observations. The final adjusted observations equal the observation plus the post-adjustment residual.

Table 8-2
Weighted Mean

System	N (meters)	σ_N	w_N	E (meters)	σ_E	w_E
DGPS	2120373.545	1.5	0.444	3617852.015	.8	1.563
Range azimuth	2120380.243	1.8	0.309	3617860.323	2.4	0.174
Weighted mean	N		σ_N	E		σ_E
	2120376.294		1.2	3617852.847		0.7

$\hat{l} = l + v$

$\hat{l}$ = adjusted observation

l = observation

v = residual

a. Functional model. The functional model relates physical or geometrical conditions to a set of observations. For example, if a surveyor measures the interior angles of a five-sided figure, the sum of these angles should add up to five hundred and forty degrees (540°). If the correct model is not determined, the adjusted observations will be in error.

b. Stochastic model. The stochastic model is the greatest advantage of the least squares procedure. In least squares adjustment, the surveyor can assign weights, variances, and covariance information to individual observations. The traverse balancing techniques and weighted means do not allow for this variability. Since observations are affected by various errors, it is essential that the proper statistical information is applied.

8-6. Observations, Blunders, and Systematic and Random Errors

a. Observations. Observations in least squares are the measurements that are to be adjusted. An adjustment is not warranted if the model is not over-determined (redundancy = 0). Observations vary due to blunders and random and systematic errors. When all blunders and systematic errors are removed from the observations, the adjustment provides the user an estimate of the "true" observation.

b. Blunders. Blunders are the result of mistakes by the user or inadvertent equipment failure. For example, an observer may misread a level rod by a tenth of a foot or a malfunctioning data recorder may cause erroneous data storage. All blunders must be removed before the least squares adjustment procedure. Blunders can be identified by scrutinizing the data before they are input in the adjustment software. Preliminary procedures like loop closures, traverse balancing, and weighted means are techniques that can identify blunders before adjustment.

c. Systematic errors. Systematic errors are the result of physical or mathematical principles. These errors must be removed before the adjustment procedure. Systematic errors are reduced or eliminated through careful measurement procedures. For example, when using DME the user should correct the distance for meteorological effects (temperature, pressure, relative humidity).

d. Random errors. Random errors are an unavoidable characteristic of the measurement process. The theories of probability are used to quantify random errors. The theory of least squares is developed under the assumption that only random errors exist within the data. If all systematic errors and blunders have been removed, the observations will differ only as the result of the random errors.

8-7. Variances, Standard Deviations, and Weights

The least squares principle incorporates the functional and stochastic models. It is essential that the correct a priori observational weights or variances are computed before the adjustment process.

a. Variance and standard deviation. The measure of variability of a set of observations is the sample (s^2) or population variance (σ^2). The greater the variance, the greater the variability of the observations. If a sample of observations has a variance of zero ($s^2 = 0$), the values of all observations are equal. Since the population mean (μ) is seldom known, the sample variance (s^2) is computed utilizing the sample mean ($\mathbf{x}$).

Sample Variance

$$s^2 = [\Sigma (x_i - \mathbf{x})^2] / (n - 1)$$

Population Variance

$$\sigma^2 = [\Sigma (x_i - \mu)^2] / N$$

where

s^2 = sample variance

σ^2 = population variance

x_i = observations (where i = 0 through n)

x = sample mean

μ = population mean

n = number of observations

N = number of elements within the population

x equals: $x = (\Sigma x_i)/n$

μ equals: $\mu = (\Sigma x_i)/N$

The sample (s) and population (σ) standard deviation are the square root of the sample or population variances. Table 8-3 shows a list of four horizontal angles using a ten-second (10") theodolite, sample mean (**x**), sample variance (s^2), standard deviation (s), and observational weight (w). If the user did not calculate the variance, the fact that the theodolite was a ten-second (10") instrument could be used as a standard deviation of the observations. However, it is advised that the surveyor calculate the variability (s^2) of a set of observations. Computing the sample variance provides a more accurate representation of the statistical nature of the observations.

b. Weights. The weight of an observation may be determined by empirical formulas, intuition, or observational analysis. The concept of weight is dependent on the a priori knowledge of the observational variance (σ^2 or s^2). The greater the observational weight, the greater the confidence in that observation. The least squares adjustment technique can accommodate absolute or relative weighting. Absolute weights are known if the observational variances have been measured or determined empirically. Relative weights are derived by intuition.

(1) Absolute weights. Absolute weights are computed empirically or through observational analysis (Table 8-3). Empirical weights are the result of experimentation or mathematical derivations. For example, DME or GPS manufacturers provide the user an equipment accuracy based on distance. These empirically derived values can be utilized to determine the variance or weight of an observation or set of observations.

Empirical Weights

$$\sigma = 5 \text{ mm} + 2 \text{ ppm} * \text{distance}$$

$$\text{mm} = \text{millimeters}$$

$$\text{ppm} = \text{parts per million}$$

If a distance of a thousand meters (1000 m) was measured between two locations, the observational weight would equal:

$$\text{ppm} = \text{millimeters (mm)/kilometers (km)}$$

$$1 \text{ km} = 1{,}000 \text{ m}$$

$$\sigma = 5 \text{ mm} + (2 \text{ mm/km}) * (1 \text{ km})$$

$$\sigma = 7 \text{ mm}$$

$$w = 1/\sigma^2$$

$$w = 0.02$$

(2) Relative weighting. Relative weighting is the result of intuition. In relative weighting, the user assigns weights based on past procedures, human factors, or physical phenomena. In level loop adjustments, relative weights (w) are considered inversely proportional to the leveled distance between stations. If the difference in elevation (de_{AB} = 0.512 m) was measured between stations A and B and the distance between the two stations was scaled from a map to be five hundred meters (500 m), the distance between the two stations could be used to compute the weight for the difference in elevation between stations A and B.

Table 8-3
Mean, Variance, Standard Deviation, and Weight

Observations	Deg	Min	Sec
1	142	20	20
2	142	20	30
3	142	20	10
4	142	20	10
Mean (**x**)	142	20	17.5
Variance(s^2)	000	01	31.6
Standard Deviation(s)	000	01	09.8
Observation Weight (w) $w = 1/s^2$			0.01

Relative Weights

$$w_{AB} \approx 1/d_{AB}$$

w_{AB} = observational weight

d_{AB} = distance between leveling stations

$$w_{AB} \approx 1/(500\ m)$$

$$w_{AB} \approx 0.002$$

8-8. Accuracy and Precision

The terms accuracy and precision are many times considered synonymous. However, they are unique, and the surveyor should use great care in how they are used to define a set of observations or coordinate values.

a. Accuracy. The accuracy of an observation is its degree of "closeness" to the true value (Figure 8-3). The RMS error statistic is often used to describe the accuracy of a set of observations. The RMS is centered about the true value and the standard deviation (σ) is centered about the mean value (**x**). The difference between the RMS and the standard deviation is the result of a bias between the measured and true value. This bias may be the result of systematic errors that were not removed prior to adjustment.

$$e = m - t$$

$$RMS = (\Sigma e^2/n)^{0.5}$$

where

Σe^2 = summation of the observational errors

m = measured value

t = true value

n = number of observations

RMS = root mean square error

b. Precision. The precision of an observation is its degree of closeness to the mean value (Figure 8-3). The variance or standard deviation is used to determine the precision of a set of observations.

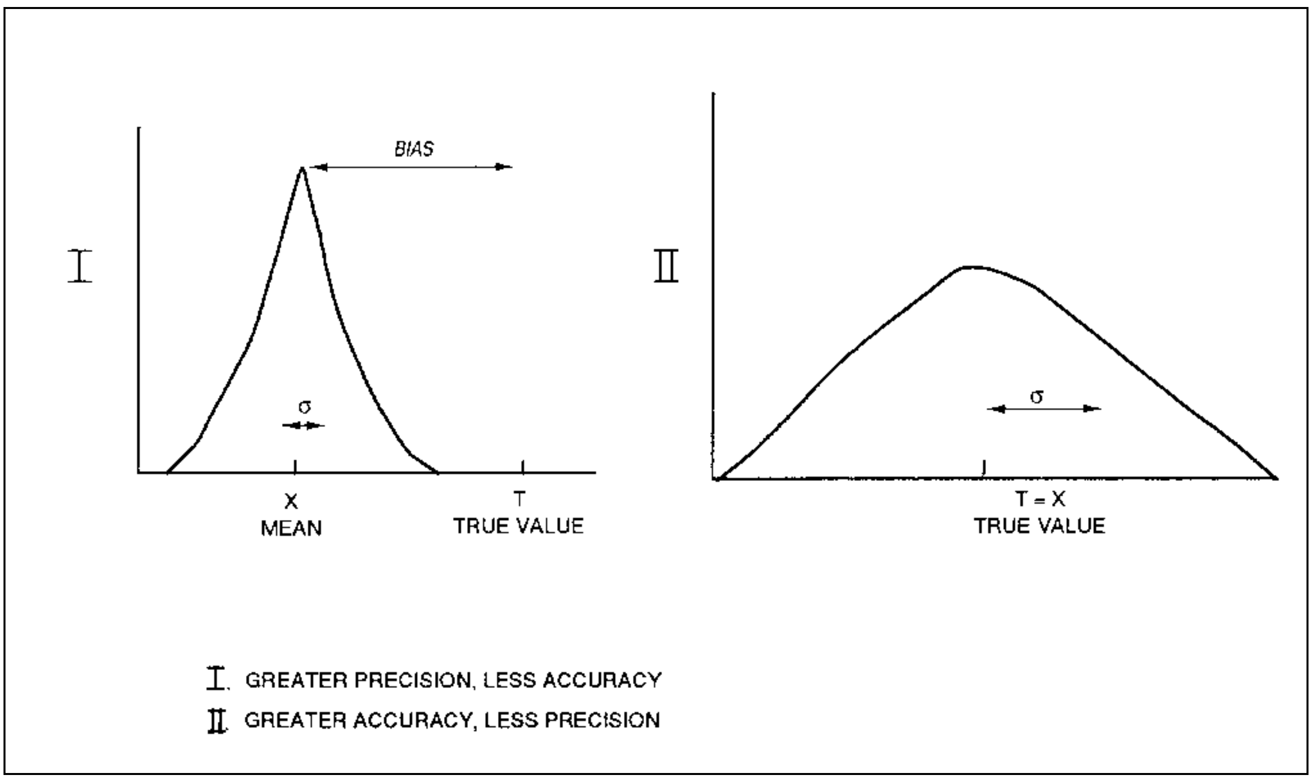

Figure 8-3. Precision and accuracy

8-9. Least Squares Adjustment Techniques

The user can employ various techniques in the adjustment of data using the least squares principle. The technique employed is dependent on the adjustment model, computational capability (computer resources), and the requirement of the survey. The reader should consult an introductory adjustment text to gain further understanding of the principles of adjustment theory. To fully understand the procedure of adjustment, a thorough understanding of matrix algebra and differential and integral calculus is required.

a. Adjustment model. The adjustment model consists of determining the number of observations to be adjusted (n), the minimum number of observations required to uniquely determine the functional model (n_o), and the redundancy (r). The model is determined by mathematical or physical relationships. For example, if the distances between three stations A, B, and C are to be determined (Figure 8-4a), the minimum number (n_o) of observations (distances) to fix the model are two. If the distance A to B (Figure 8-4b) was measured nine times and the distance B to C was not measured, the model could not be determined if the objective was to adjust the distance between A and C. Therefore, it is not only important to have redundant observations, but it is critical to have the correct number of observations to fix the model (n_o).

b. Observations only ($Av = f$). This method is seldom utilized because generalized software packages are difficult to develop. The method involves creating a condition or set of conditions that satisfies the functional model. Figure 8-5 shows a level loop involving three stations. Table 8-4 includes the differences in elevations and distances between stations A, B, and C. To perform a least squares adjustment for this level network, the adjustment model must be determined (n,n_o,r). The number of observations are the difference in elevations between the points (de_{ab},de_{bc},de_{ca}). If one station has a known elevation, two observations or difference in elevations are required to fix the adjustment model (n_o). If two stations have known elevations, one observation or difference in elevations is required to fix the adjustment model (n_o). In general, the minimum number of observations required to uniquely determine a level loop equals the number of stations in the loop minus the number of known stations. The redundancy for the model equals the number of observations minus the minimum number of observations to fix the model ($r = n - n_o$). The observations-only technique involves the use of condition

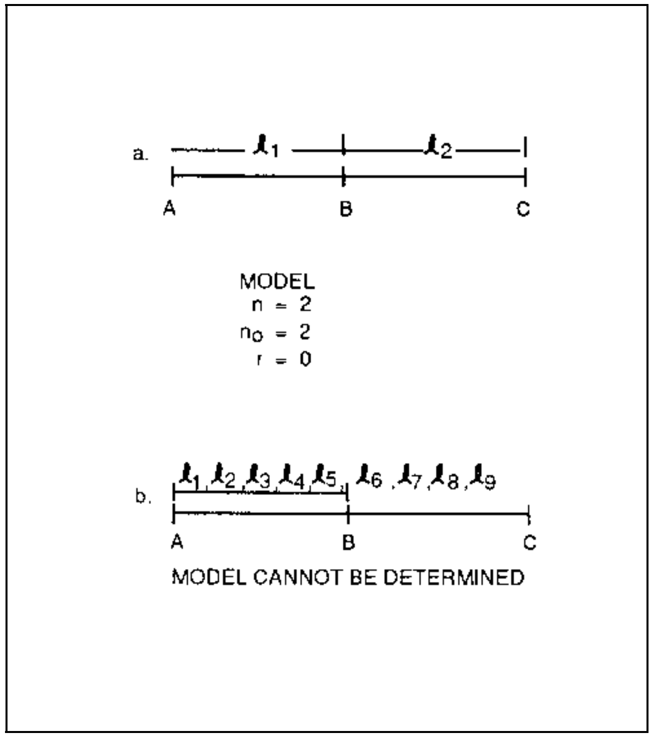

Figure 8-4. Adjustment model

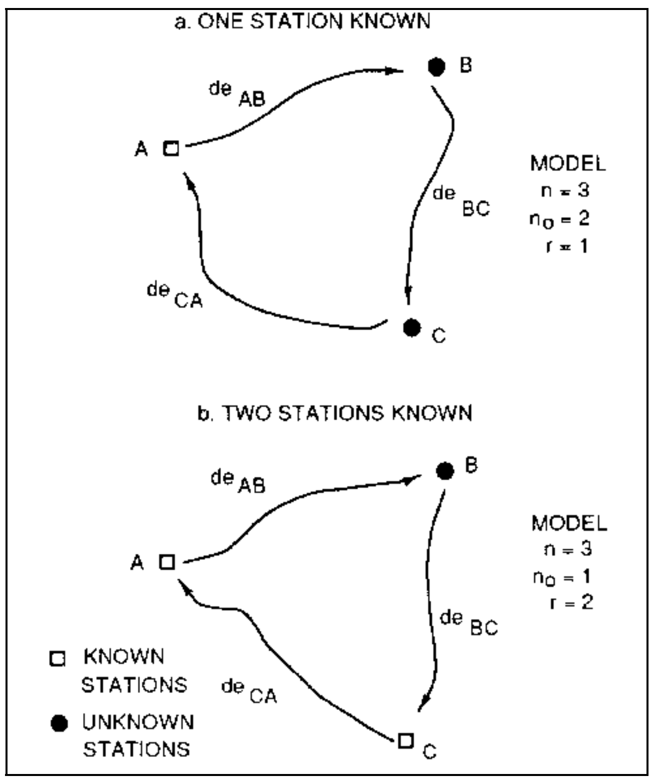

Figure 8-5. Level loop

Table 8-4
Level Loop

Known Elevation A = 232.150 m Adjusted Elevation B = ? Adjusted Elevation C = ?			
From	To	Difference in Elevation (DE) (meters)	Distance (meters)
A	B	-10.234	500
B	C	2.324	1200
C	A	7.821	1000

equations. In this example the elevation of one station (A) is considered known, the redundancy (r) equals one, and the condition equation equals the summation of the elevation differences from A, B, and C ($de_{ab} + de_{bc} + de_{ca} = 0$). In the observations-only technique, the number of condition equation equals the redundancy of the model.

Condition Equation

$$C = de_{ab} + de_{bc} + de_{ca} = 0$$

Step 1. Determine the number of observation (n), the minimum number of observations (n_o), and the redundancy ($r = n - n_o$) that satisfies the functional model.

Adjustment Model

$$n = 3$$

$$n_o = 2$$

$$r = 1$$

Compute the cofactor matrix (Q) which is the inverse of the weight matrix. In the level loop example the weight matrix is derived using the distance between leveling stations. The weight of each of observation is inversely proportional to the leveled distance. The dimensions of the cofactor and weight matrix are n x n (3 x 3).

where

n = number of observations

$$Q = W^{-1}$$

Q = cofactor matrix

W = weight matrix

$$W = \begin{bmatrix} \frac{1}{500} & 0.0 & 0.0 \\ 0.0 & \frac{1}{1200} & 0.0 \\ 0.0 & 0.0 & \frac{1}{1000} \end{bmatrix}$$

$$Q = \begin{pmatrix} 500 & 0 & 0 \\ 0 & 1200 & 0 \\ 0 & 0 & 1000 \end{pmatrix}$$

Step 2. Formulate the design matrix **A** and the misclosure vector **f**. The design matrix and misclosure vector have dimensions c x n (1 x 3) and c x 1 (1 x 1).

where

c = number of condition equations

n = number of observations

If the condition equations are linear, the elements of the design matrix (A) equal the coefficients of each observation in the condition equation. If the condition equations are nonlinear, they must be linearized using a Taylor series expansion. The condition equations for the level loop example are linear. The misclosure vector (f) equals the negative of the condition equations.

$$C = de_{ab} + de_{bc} + de_{ca} = 0$$

$$A = [\ 1\ 1\ 1]$$

$$f = [-de_{ab}\ - de_{bc}\ - de_{ca}]$$

$$f = [\ 10.234\ - 2.324 - 7.821]$$

$$f = [\ 0.089]$$

Step 3. The observational residuals are computed by the following matrix multiplications.

$k = (AQA^t)^{-1}f$

$k = [3.2963E^{-5}]$

$v = QA^t k$

$v = [\ 0.016\ 0.040\ 0.033]^t$

Step 4. The adjusted observations ($l^\wedge$) are computed by adding the post-adjustment residuals (v) to the measured observations.

$$de^\wedge = de + v$$

$$de^\wedge = \begin{pmatrix} -10.234 \\ 2.324 \\ 7.821 \end{pmatrix} + \begin{pmatrix} 0.016 \\ 0.040 \\ 0.033 \end{pmatrix}$$

$$de^\wedge = \begin{pmatrix} -10.218 \\ 2.364 \\ 7.854 \end{pmatrix}$$

The adjusted observations ($de^\wedge$) are utilized to determine if the condition was satisfied. Computational errors or an incorrect functional model can cause the adjusted observations not to satisfy the condition equations.

$$C = de_{ab} + de_{bc} + de_{ca} = 0$$

$$C = -10.218 + 2.364 + 7.854 = 0$$

Step 5. The estimated elevations of stations **B** and **C** are computed using the known elevation of station **A** and the adjusted differences in elevation ($de^\wedge$).

Elevation Estimates

$B = A + de_{ab}$

$B = 232.150\ m + (-10.218\ m)$

$B = 221.932\ m$

$C = A + de_{ac}$

$C = 232.150\ m + (-7.854\ m)$

$C = 224.296\ m$

c. Indirect observations ($v + B\Delta = f$). The indirect method includes observations and user-defined parameters. Observation equations are developed using parameters in terms of one observation. This method is commonly employed by commercial software packages. Figure 8-6 represents a survey where three ranges (distances) were measured from known survey control stations (A, B, and C) to a hydrographic vessel (D). The unknown position of the vessel was computed using the method of indirect observations. A minimum of two ranges (distances) (n_o = 2) are required to determine the unknown location of the vessel. The total number of observations is three (n = 3) and the redundancy of the system equals one (r = 1). The following steps outline the method of adjustment using indirect observations. Table 8-5 contains the observations, standard deviations, and coordinate information for the problem illustrated in Figure 8-6.

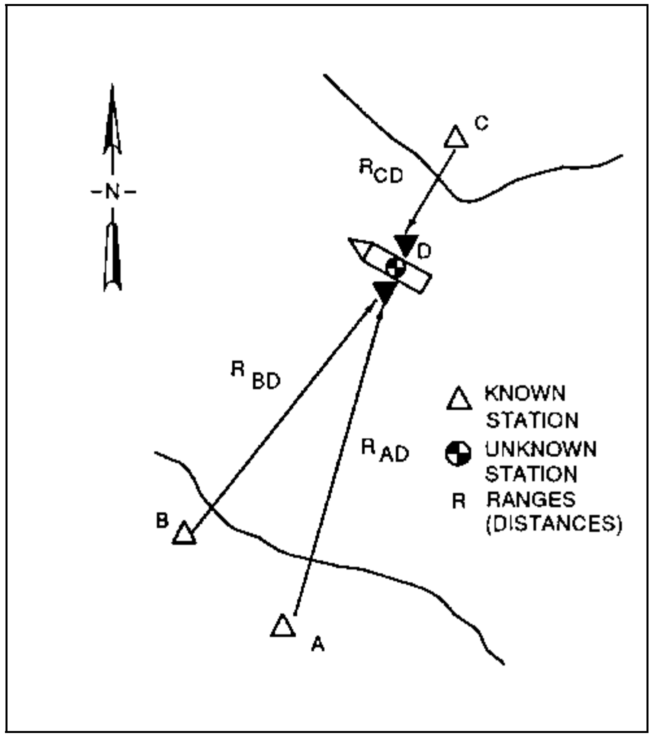

Figure 8-6. Intersection problem

Step 1. Determine the number of observations (n), the minimum number of observations (n_o), and the redundancy ($r = n - n_o$) that satisfies the functional model.

Adjustment Model

$n = 3$

$n_o = 2$

$r = 1$

Table 8-5
Intersection Example

Station	N (meters)	E (meters)	Range (R) to Station D (meters)	Range Standard Deviation (σ_R) (meters)
A	2111613.416	3616819.580	8562.825	.500
B	2117118.247	3610160.570	8238.020	.600
C	2120373.830	3617852.048	258.714	0.5

The weight matrix is (w) computed by using the stochastic properties of the measured ranges (σ_R). The weight is considered to be inversely proportional to the variance.

$$W = \begin{bmatrix} \frac{1}{(\sigma^2)_{ad}} & 0 & 0 \\ 0 & \frac{1}{(\sigma^2)_{ad}} & 0 \\ 0 & 0 & \frac{1}{(\sigma^2)_{ad}} \end{bmatrix} \quad W = \begin{pmatrix} 4.0 & 0 & 0 \\ 0 & 3.0 & 0 \\ 0 & 0 & 4.0 \end{pmatrix}$$

Step 2. In the method of indirect observations ($v + B\Delta = f$) the number of condition or observation equations is equal to the total number of observations (n = 3). The observation equations (F_n) are developed such that they satisfy the functional model. The distance equation satisfies the functional model for the example in Figure 8-6.

Observation Equations

$$F_1 = R_{AD} - [\ (N_D - N_A)^2 + (E_D - E_A)^2\ ]^{0.5} = 0$$

$$F_2 = R_{BD} - [\ (N_D - N_B)^2 + (E_D - E_B)^2\ ]^{0.5} = 0$$

$$F_3 = R_{CD} - [\ (N_D - N_C)^2 + (E_D - E_C)^2\ ]^{0.5} = 0$$

Each observation equation (F_1,F_2,F_3) must be linearized with respect to the unknown parameters (N_D, E_D) using a Taylor series expansion. The design matrix (B) is comprised of the linearized observation equations. The design matrix has dimensions n x n_o (3 x 2). The nonlinearity of the problem requires that initial approximations are computed for the unknown parameters. The initial approximations (N^o_D, E^o_D) for the hydrographic vessel were computed from stations **A** and **B** using the technique of range-range intersection.

$$N^o_D = 2{,}120{,}115 \text{ meters}$$

$$E^o_D = 3{,}617{,}833 \text{ meters}$$

Design Matrix

$$B = \begin{bmatrix} \frac{\delta F_1}{\delta N_D} & \frac{\delta F_1}{\delta E_D} \\ \frac{\delta F_2}{\delta N_D} & \frac{\delta F_2}{\delta E_d} \\ \frac{\delta F_3}{\delta N_D} & \frac{\delta F_3}{\delta E_D} \end{bmatrix} \quad \frac{\delta F}{\delta N_D} = \frac{N_A - N_d}{\sqrt{(N_D - N_A)^2 + (E_D\ E_A)^2}} \quad \frac{\delta F}{\delta E_D} = \frac{E_A - E_D}{\sqrt{(N_D - N_A)^2 + (E_D - E_A)^2}}$$

The numerical elements of the design matrix (B) are computed by evaluating the partial derivatives ($\delta F_n/\delta N_D$, $\delta F_n/\delta E_D$) of the observation equations at the initial approximations (N^o_D, E^o_D) of the unknown parameters. The misclosure vector equals the negative of the observation equations. The misclosure vector (f) has dimensions n x 1 (3 x 1).

Misclosure Vector

$$f = \begin{matrix} F_1 = [\ (N_D - N_A)^2 + (E_D - E_A)^2\ ]^{0.5} - R_{AD} \\ F_2 = [\ (N_D - N_B)^2 + (E_D - E_B)^2\ ]^{0.5} - R_{BD} \\ F_3 = [\ (N_D - N_C)^2 + (E_D - E_C)^2\ ]^{0.5} - R_{CD} \end{matrix}$$

$$B: = \begin{pmatrix} -.9930 & -.1184 \\ -.3638 & -.9315 \\ .9973 & .0734 \end{pmatrix}$$

$$f: = \begin{pmatrix} -.1.053 \\ -1.109 \\ -0.184 \end{pmatrix}$$

Step 3. The estimates of the unknown parameters (Δ) and the residuals are computed by the following matrix manipulations. Since the problem is non-linear the algorithm must be repeated (iterated) until the solution converges. This example required two iterations to converge. The number of iterations is a function of data quality, initial approximations, and the functional model. The condition that the sum of the squares of the residuals

($v^t wv$) equals a minimum (ϕ) is utilized as a convergence criterion. The parameter vector (Δ) has dimensions n_o x 1 (2 x 1). In nonlinear problems the parameter vector must be added to the initial approximations to obtain the adjusted estimates.

Iteration 1

$$\Delta = (B^t WB)^{-1} * B^t Wf$$

$$\Delta := \begin{pmatrix} 0.323 \\ 1.107 \end{pmatrix}$$

$N^1_D = N^o_D + \Delta(1,1) = 2,120,115 \text{ m} + 0.323 \text{ m} = 2,120,115.323 \text{ m}$

$E^1_D = E^o_D + \Delta(2,1) = 3,617,833 \text{ m} + 1.107 \text{ m} = 3,617,834.107 \text{ m}$

$$v = f - B\Delta$$

$$v := \begin{pmatrix} -0.601 \\ 0.040 \\ -5.87 \end{pmatrix}$$

$$\phi_1 = v^t Wv = 2.83$$

Iteration 2

$$B := \begin{pmatrix} -.9930 & -.1185 \\ -.3638 & -.9315 \\ .9976 & .0694 \end{pmatrix}$$

$$f := \begin{pmatrix} -0.601 \\ 0.04 \\ -0.584 \end{pmatrix}$$

$$\Delta := \begin{pmatrix} 0.001 \\ 0.004 \end{pmatrix}$$

$$\phi_2 = 2.81$$

If $|\phi_1 - \phi_2|/\phi_1 < 0.01$ terminate adjustment

$(2.83 - 2.81)/2.83 = 0.007$

$0.007 < 0.01$

Final Adjusted Coordinates of Station D

$N_D = N^1 + \Delta(1,1) = 2,120,115.323 + 0.001 = 2,120,115.324 \text{ m}$

$E_D = E^1 + \Delta(2,1) = 3,617,834.107 + 0.004 + 3,617,834.111 \text{ m}$

d. General least squares ($Av + B\Delta = f$). This is the general case for the observations-only and indirect method. In some problems, equations that satisfy the functional model using the observations-only or indirect observations method may be difficult to develop. In the general case the condition equations can contain both parameters and multiple observations. The algorithm for this technique can be obtained by consulting an introductory adjustment textbook.

8-10. Error Analysis

After the adjustment procedure, the data are examined for observational blunders. Blunders and systematic errors should be identified and removed before the adjustment. However, erroneous observations are not always recognized before the initial adjustment procedure. To ensure that the final estimates are "free" of blunders, a blunder detection scheme should be implemented. The common blunder detection techniques are the global variance test, data snooping method, tau test, and robust estimation. Commercial software packages commonly employ the global and tau tests for the determination of blunders.

a. Statistical inference. Statistical inference involves the statement of a hypothesis. Statistical inference is most commonly utilized to identify observational blunders within the adjustment. The adjusted data are tested or compared to determine if the hypothesis is satisfied.

Hypothesis

H_o: Adjustment estimates are "free" of blunders

H_a Blunders

Four outcomes are possible from hypothesis testing:

(1) Select the null hypothesis (H_o), when the null hypothesis is true (correct decision).

(2) Select the alternative hypothesis (H_a), when the alternative hypothesis is true (correct decision).

(3) Select the alternative hypothesis (H_a), when in fact the null hypothesis (H_o) is true (type I error).

(4) Select the null hypothesis (H_o), when in fact the alternative hypothesis (H_a) is true (type II error).

The significance value (α) is dependent on the probability of committing a type I error. The probability of committing a type II error (1- β) is the result of accepting the null hypothesis when in fact the alternative hypothesis is true. The user must determine which error (type I or type II) will be the most costly. A significance level that is very large (α= 0.1) would decrease the confidence level (1-α). The smaller confidence level could result in the possible rejection of "good" observations. Therefore, if a system had limited redundancy, a smaller significance level may be warranted (α = 0.005).

b. Global variance test. Some software packages provide the user the opportunity to input a significance level or probability value. The a posteriori reference variance is one of the results of adjustment. If the a priori reference variance is known, the a posteriori variance is tested to determine if it is consistent with the a priori variance. The global test is applicable only when absolute weights were utilized in the adjustment. A two-tailed or upper tail test is constructed to test the variances.

A Posteriori Reference Variance

$$\sigma^2 = (v^t wv)/r$$

v^t = residual transposed

v = residual

r = redundancy

Two-Tailed Test

$$H_o : \sigma^2 = \sigma_o^2$$

$$H_a : \sigma^2 \neq \sigma_o^2$$

The two-tailed test fails if

$$X_r^2 < X^2_{\alpha/2,r} \text{ or } X_r^2 > X^2_{1-\alpha/2,r}.$$

Upper Tail Test

$$H_o : \sigma^2 = \sigma_o^2$$

$$H_a : \sigma^2 > \sigma_o^2$$

The upper tail test fails if $X_r^2 > X^2_{\alpha,r}$

$$X_r^2 = v^t wv/\sigma_o^2 = r\sigma^2/\sigma_o^2$$

$X^2_{\alpha,r}$, $X^2_{\alpha/2,r}$, $X^2_{1-\alpha/2,r}$ are critical values that are calculated from the chi square distribution based on a significance level alpha (α) and redundancy (r).

The failure of the global test suggests the possibility of a blunder; however, it does not identify the location of the blunder. Failure of the global test may also be the result of incorrect weighting or an incorrect adjustment model. The global test should be used in conjunction with the data snooping method or robust estimation. If an empirical method is used to determine the weights, the user must determine if the computed values are realistic. Otherwise, the global test may fail due to incorrect a priori weights. If the global test is rejected, the data require inspection for blunders or incorrect weighting. If the a posteriori reference variance passes the global test, it does not guarantee the absence of blunders within the adjustment. Therefore, the global test should only be used in conjunction with another analysis method.

c. Data snooping. The data snooping method requires the user to know the a priori reference variance. The method is very effective when only one blunder is in the network. The technique of data snooping sequentially tests each standardized residual within the adjustment and determines if it exceeds a defined rejection threshold. The standardized residual (v') is defined as the observational residual divided by the residual's standard deviation. The rejection threshold is computed based on a given significance level (α_o) using the Fisher distribution.

Standardized Residual

$$v' = v/\sigma_v$$

v' = standardized residual

σ_v = $(\sigma_o) * (q_{vi})^{0.5}$

σ_o = a priori reference standard deviation

q_{vi} = the ith diagonal element of the residual cofactor matrix (Q_{vv})

Q_{vv} = $Q - B(B^t WB)^{-1}B^t$

Data Snooping Rejection Criteria

$$|v'| > (F_{(1-\alpha o)},1,oo)^{0.5}$$

$$\text{If } |v'| > 3.29 \quad @ \ \alpha_o = 0.001$$

If the standardized residual exceeds the rejection threshold, it is considered an outlier. The suggested significance value for the data snooping technique is α_o = 0.001. The rejection threshold corresponding to a significance value of 0.001 is 3.29. Therefore, the absolute value of all standardized residuals (v') that exceed 3.29 are identified as possible blunders. The data snooping technique is a univariate test that is very effective when only one blunder is in the network. Therefore, only the standardized residual with the greatest value that exceeds the rejection threshold is removed from the observations. After removal of the possible blunder, the adjustment is re-computed and the standardized residuals are tested for additional blunders. The procedure is continued until all blunders have been removed from the network.

d. Tau test. The tau test is used when the a priori reference variance is unknown. The tau test utilizes the tau distribution to compute the critical values. The tau ($\underline{t}$) distribution can be derived from the **t** (student) distribution using the following formula:

Tau Distribution

$$\underline{t} = [(r)^{0.5} * \mathbf{t}_{r-1}]/[(r-1 + \mathbf{t}^2_{r-1})^{0.5}]$$

r = redundancy

t = critical value from the **t** (student) distribution based on a significance level (α)

The rejection threshold is computed given a significance value (α) and the adjustment redundancy (r). The threshold is compared to the standardized residual. Since the a priori reference variance is not known, the standardized residual is computed using the a posteriori reference variance (σ^2). If the standardized residual exceeds a rejection threshold based on the tau distribution, the observation is flagged as a blunder. The tau rejection thresholds are interpolated from tables or generated from computer subroutines. After removal of the blunder, the adjustment is re-computed and the standardized residuals are tested for additional blunders. The procedure is continued until all blunders have been removed from the network.

Standardized Residual

$$v' = v/\sigma\ (q_{vi})^{0.5}$$

v' = standardized residual

σ = a posteriori standard deviation

q_{vi} = residual of the ith cofactor element

Tau Test Rejection Threshold

$$\text{If } v' > \underline{t}$$

Both the data snooping technique and tau test require the computation of the residual cofactor matrix (Q_{vv}). The determination of the residual cofactor matrix is computationally time consuming. To alleviate this time inefficiency, the diagonal components can be computed. This is only viable when the observational weight matrix is block diagonal. Another approach is the replacement of the standard deviation of the residual (σ_v) with the observational standard deviation (σ_l). The residual standard deviation is smaller than the observational standard deviation.

$$|v|/\sigma_l < |v|/\sigma_v$$

Therefore, an observation with a blunder may not be flagged as a possible erroneous measurement because the standardized residual will be smaller. If the observational standard deviation (σ_l) is substituted for the residual standard deviation (σ_v), it is recommended that the significance value (α) be increased. Increasing the significance value (α) will cause a decrease in the confidence level.

e. Robust estimation. The technique of robust estimation does not depend on the residual cofactor matrix (Q_{vv}) or the significance value (α). If the a priori reference variance is known, it is recommended that a global variance test be performed for the first adjustment iteration. The global test provides additional information on the presence of blunders or inaccuracies in the adjustment model or a priori weights. The robust estimation procedure changes the observational weights during each iteration for those observations that exceed a predefined threshold. The absolute value of the residuals divided by the observational standard deviations ($|v|/\sigma_l$) are sequentially analyzed to determine if they exceed a rejection threshold. If the residual exceeds the rejection threshold, the observational weight is reduced using a decreasing weight function. In essence, this technique removes observations with large residuals from the adjustment. The adjustment is continued until the solution converges. When the adjustment is completed all blunders should have been removed. The following decreasing weight functions are commonly utilized in the technique of robust estimation.

Weighting Function 1

All iterations:

$$w^{i+1} = w^{i}\exp^{-|vi/\sigma i|} \qquad \text{if } |v| \succ 3$$

$$w^{i+1} = w^{i} \qquad \text{if } |v| \succ 3$$

Weighting Function 2

Iteration 1

$$w^{i+1} = w^{i}$$

Iteration 2 & 3

$$w^{i+1} = w^{i}(\exp[(|v_i/\sigma_i|)^{4.4}])^{0.05} \qquad \text{if } |v| \succ 3$$

$$w^{i+1} = w^{i} \qquad \text{if } |v| \succ 3$$

following iterations:

$$w^{i+1} = (\exp[-(|v_i/\sigma_i|^{3}])^{0.05}$$

w^{i} = the weight of the ith observation

exp = exponential function

v_i = observational residual

σ_i = observational standard deviation

8-11. Interpretation and Analysis of Adjustment Results

The interpretation of adjustment results does not require knowledge of adjustment theory or advanced mathematics. The following section provides various “thumb rules” that can be utilized to determine the quality and reliability of the adjusted data. Although many software routines provide error analysis or blunder detection options the user must carefully interpret the results of these techniques. Data should not be rejected solely on the results provided by these packages.

a. Input parameters.

(1) Significance level (α). If the adjustment software provides the flexibility of inputting a significance level (α) the user should choose a value of 0.05. This value minimizes the probability of committing a type I and type II error.

(2) Initial coordinate estimates. Some programs require the user to input initial coordinate estimates. It is imperative that these values be realistic. If the initial estimates are erroneous, the adjustment may not converge to the correct solution. Techniques like traverse balancing and the weighted mean should be utilized to determine initial coordinates.

(3) Data input. All data must be entered in the correct linear and angular units. Never input one variable (coordinates) in feet and another variable (distances) in meters. Most software packages cannot accommodate mismatched units.

(4) Checking. Before the adjustment is computed, all field, office, and computer generated input should be checked by two individuals for blunders. Also, all known systematic errors should be removed from the data (i.e., meteorological data, collimation, and leveling corrections).

b. Output. The majority of the manufacturers’ output the a posteriori reference variance, standardized residuals, and error ellipse information. Many software routines have blunder detection schemes that “flag” and remove observations that are possible blunders. These techniques of blunder detection are based on statistical tests like tau or data snooping. The drawback with these methods is they are many times unreliable. Therefore, the user must develop a “horse sense” in the determination and identification of blunders.

(1) Global test. The global test is utilized to determine blunders. However, if absolute weights were not used in the adjustment the results of the test are meaningless.

(2) Standardized residuals. The standardized residuals are an excellent indicator in determining blunders. After all blunders have been identified and removed using the manufacturers’ software the output should be examined for additional blunders that were not located.

(3) Error ellipse.

f. Error ellipse. The error ellipse provides a representation of the precision of the adjusted parameters and observations. The error ellipse consists of semimajor (a) and semiminor axes (b). The semimajor (a) and semiminor (b) axes are precision estimates (σ) of the adjusted parameters. The error ellipse can be utilized to determine the absolute or relative precision of parameters.

8-12. Contract Monitoring

a. Recommendations. Contracts involving surveys that are small in areal extent and require low accuracy survey control (1:5,000) may not warrant a least squares adjustment. In general, Fourth-Order surveys (1:5,000) do not warrant adjustment. A compass rule adjustment or observational mean will suffice. All control surveys that are to be incorporated into the NGRS shall be performed and adjusted using the guidelines established by the NGS.

b. Required submittal documents. The contracting officer should require the contractor to supply the final adjustment for each project. The contractor should be required to supply a list containing any observations that were removed due to blunders. The contractor must provide the Corps with a detailed analysis explaining the methodology performed in the adjustment, assumptions, and possible error sources.